



Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji



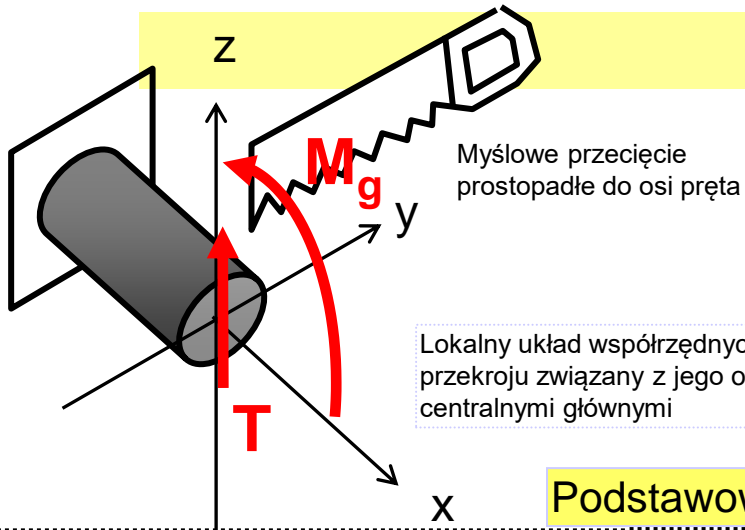
Wykład 9

Pręty zginane – belki

Wyznaczanie rozkładu sił
wewnętrznych



Zginanie proste



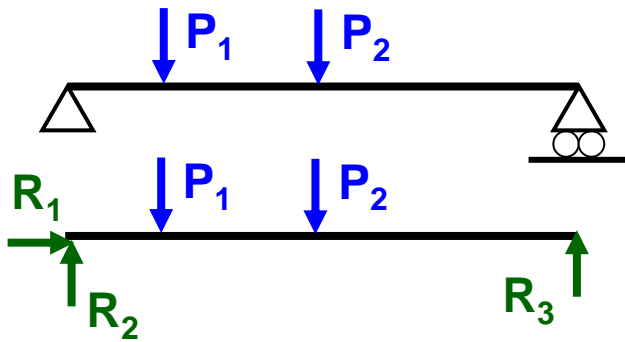
Występują składowe wysiłki przekroju: T – Siła tnąca
 M_g – Moment gnący

Jeśli $T = 0$ to mamy zginanie czyste

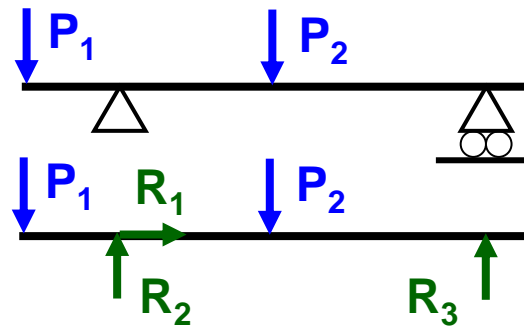
Jeśli $T \neq 0$ to mamy zginanie poprzeczne

Podstawowe schematy belek statycznie wyznaczalnych:

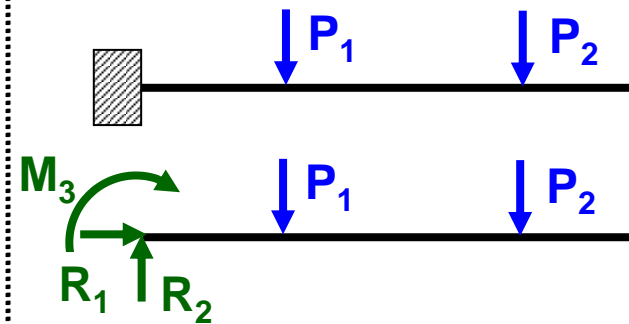
a) dwupodporowa



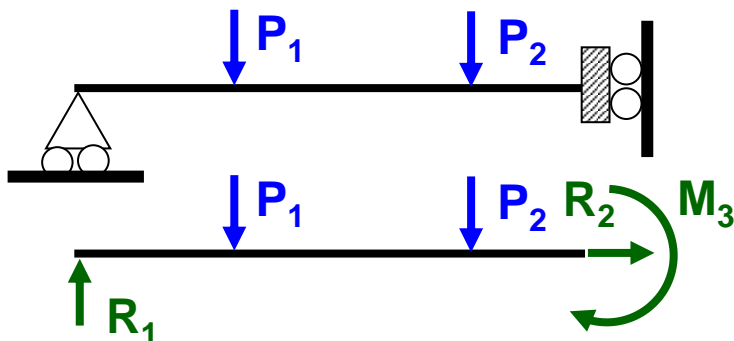
b) dwupodporowa z wysięgnikiem



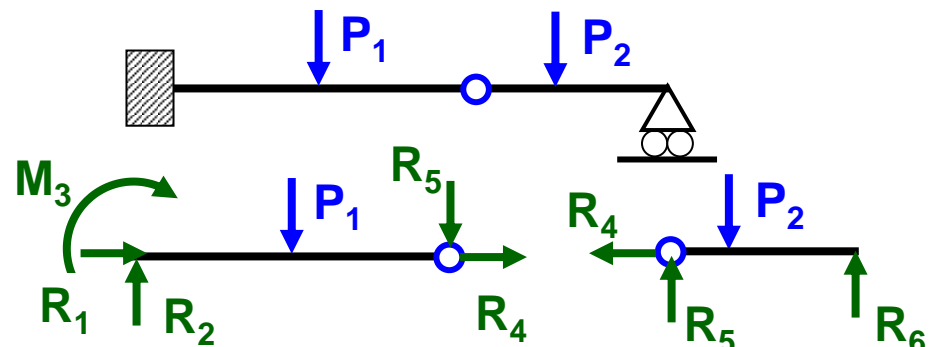
c) wspornikowa



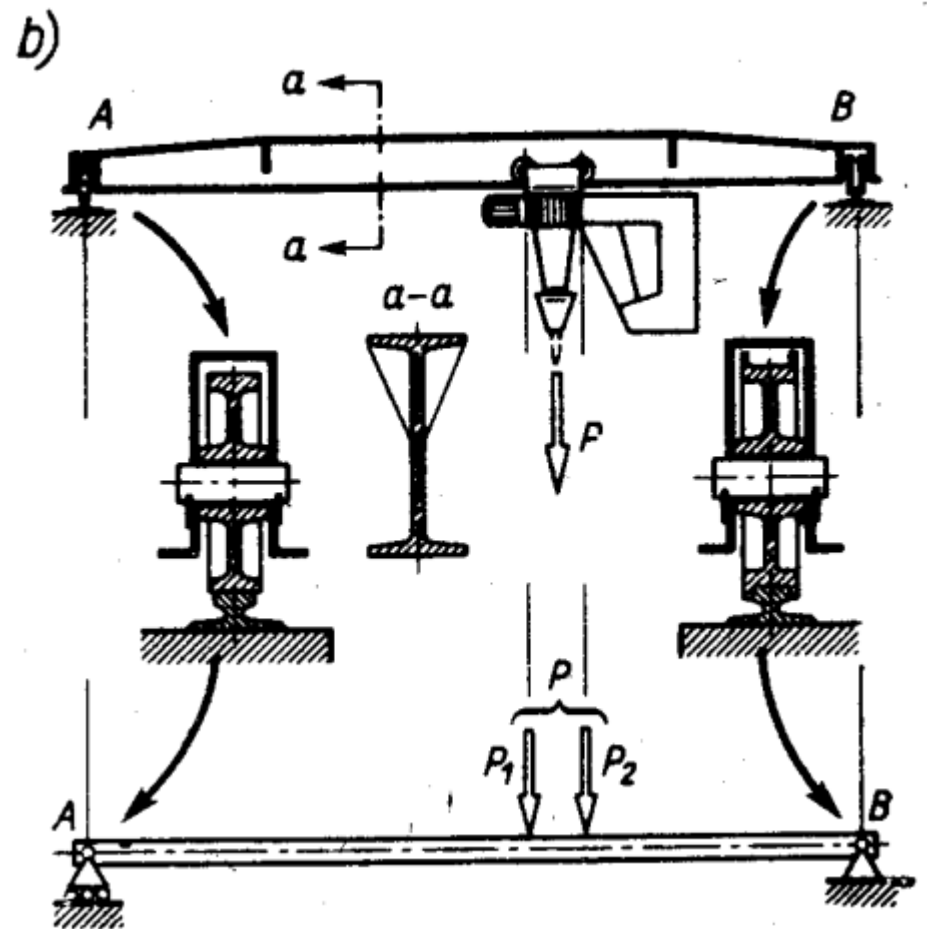
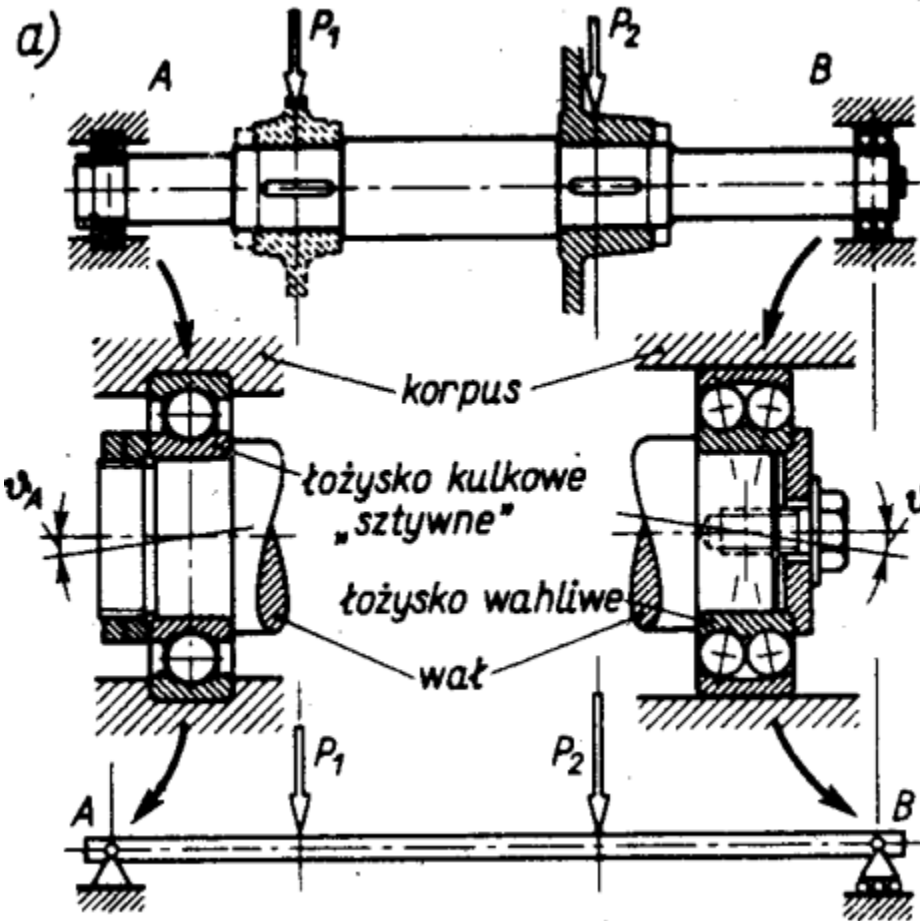
d) z wózkiem odbierającym moment



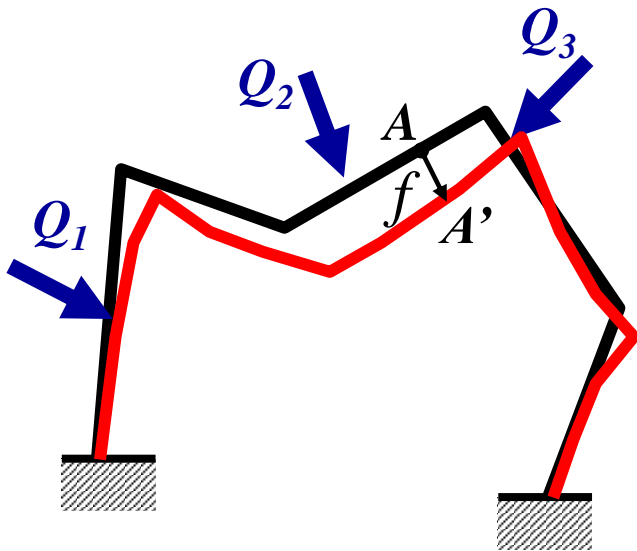
e) z przegubem



Przykłady tworzenia schematu obliczeniowego



Zasada superpozycji



Konstrukcja liniowa : Przemieszczenia i naprężenia są liniowymi funkcjami obciążeń

$$f = \sum_{j=1}^n \alpha_j Q_j$$

$$\sigma = \sum_{j=1}^n \beta_j Q_j$$

α_j, β_j - współczynniki liniowe

Warunki: 1) Materiał liniowo-sprężysty
2) Odkształcenia są małe
3) Przemieszczenia są małe

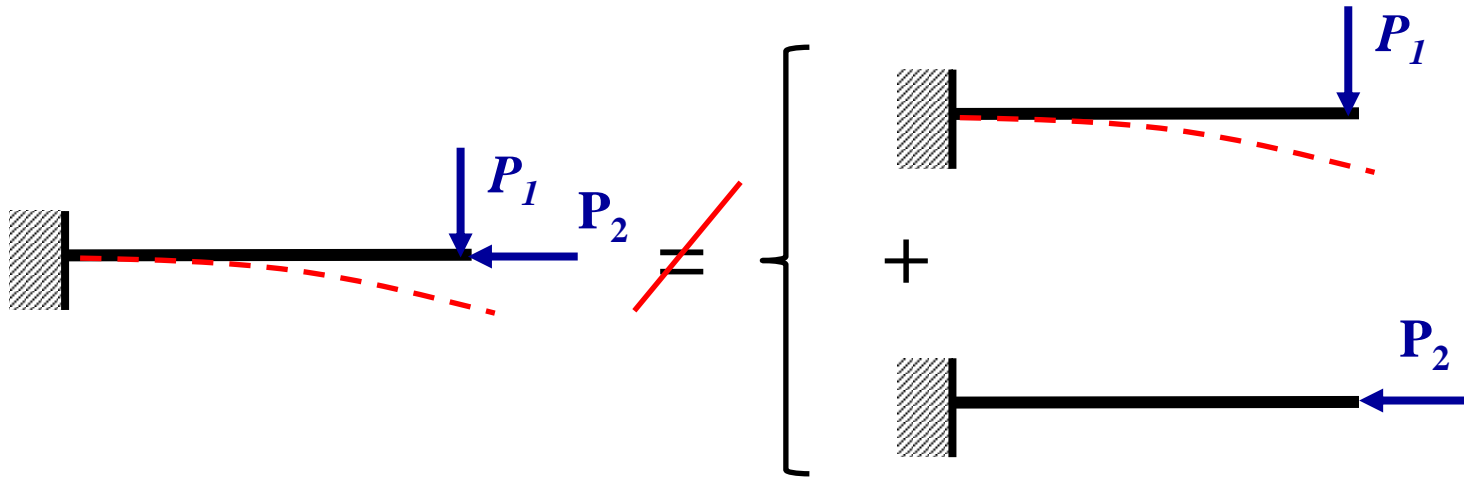
Zasada zeszywnienia (wymiary początkowe)

Zasada superpozycji (niezależności działania obciążeń)

W przypadku, gdy na konstrukcję liniową działa złożony układ obciążeń, to skutek działania tego układu jest równy sumie skutków obciążeń składowych

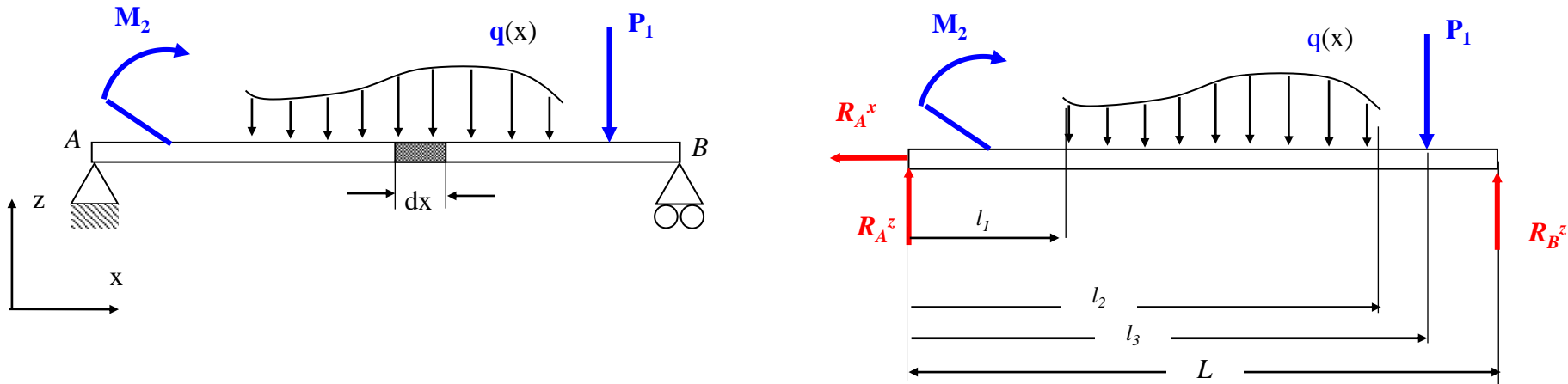


Nieraz nie można zastosować zasady superpozycji



Wyznaczanie składowych wysiłku przekroju w prostym zginaniu

Zginanie to taki przypadek obciążenia pręta, w którym występuje siła tnąca T i moment gnący Mg



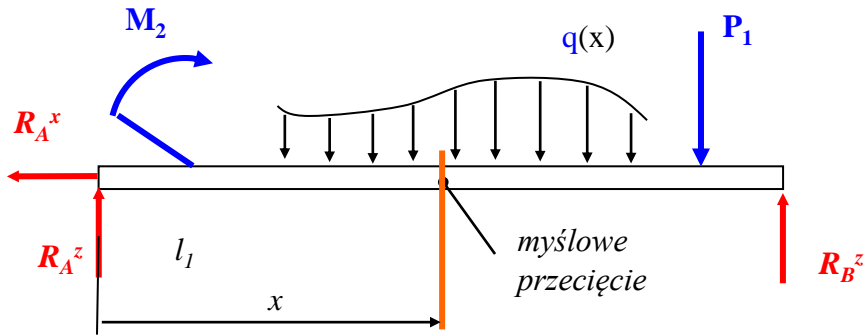
Zwykle zadanie zaczynamy od uwolnienia belki od więzów i wyznaczenia reakcji:

$$\sum F_x = 0: \quad R_A^x = 0$$

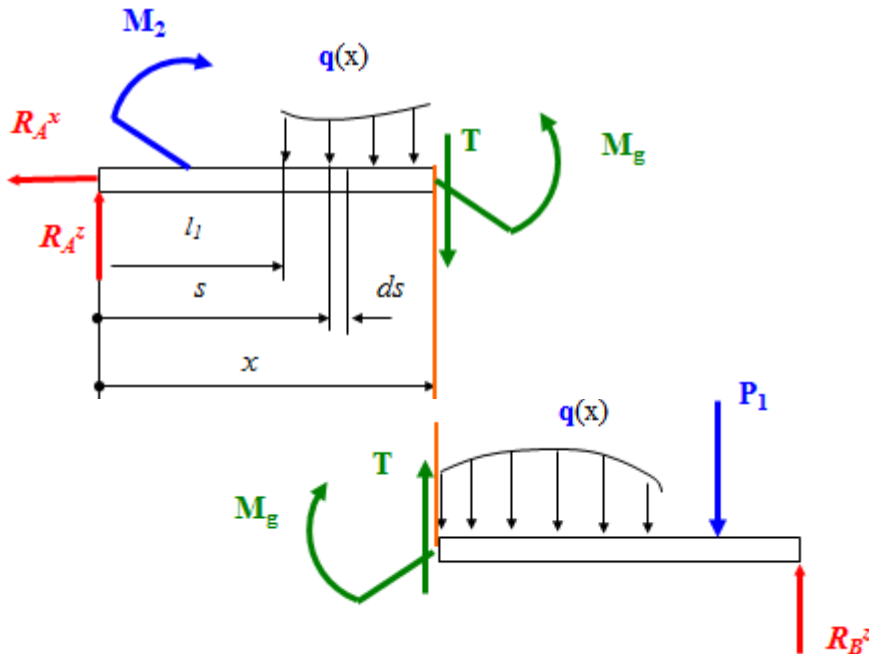
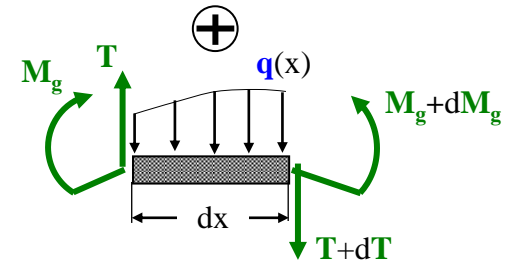
$$\sum M_A = 0: \quad R_B^z \cdot L - P_1 \cdot l_3 - \int_{l_1}^{l_2} q(x) \cdot x \, dx - M_2 = 0$$

$$\sum F_z = 0: \quad R_A^z + R_B^z - P_1 - \int_{l_1}^{l_2} q(x) \, dx = 0$$

Wyznaczanie składowych wysiłku przekroju w prostym zginaniu



Konwencja dodatnich znaków



Poszukiwane funkcje $M_g(x)$ i $T(x)$ możemy wyznaczyć z warunku równowagi myślowo wyciętej części (np. lewej):

$$\sum F_z = 0: R_A^z - T(x) - \int_{l_1}^x q(s) ds = 0$$

$$\sum M_{pp} = 0:$$

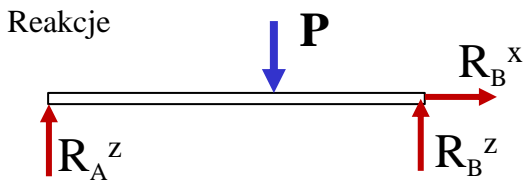
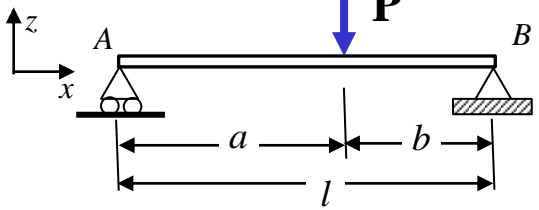
$$-R_A^z \cdot x + \int_{l_1}^x q(s) \cdot (x-s) ds - M_2 + M_g(x) = 0$$

Warto pamiętać, że pomiędzy funkcjami $M_g(x)$, $T(x)$ i $q(x)$ zachodzą związki:

$$T(x) = \frac{dM_g(x)}{dx}$$

$$q(x) = -\frac{dT(x)}{dx}$$

Zadanie 1



Równania równowagi:

$$\sum F_x = 0 \rightarrow R_B^x = 0$$

$$\sum M_A = 0 \rightarrow R_B^z \cdot l - P \cdot a = 0$$

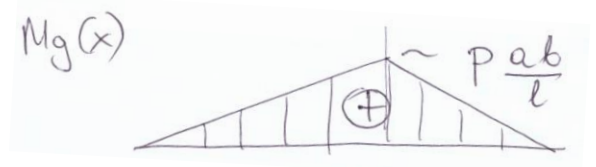
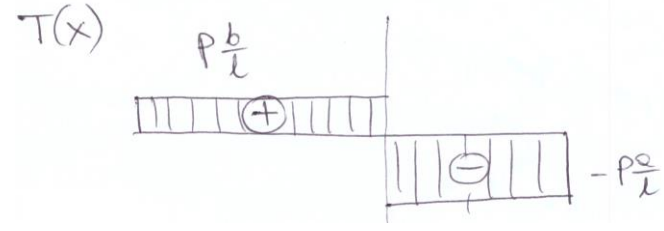
$$R_B^z = P \cdot a / l$$

$$\sum M_B = 0 \rightarrow -R_A^z \cdot l + P \cdot b = 0$$

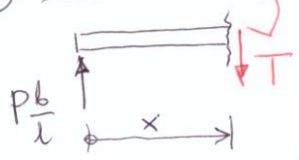
$$R_A^z = P \cdot b / l$$

Wyniki:

$$T = \frac{dM_g}{dx}$$

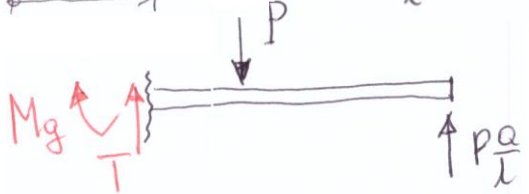


(I) $x \in (0, a)$



r-wo r-gi: $\sum F_z = 0$:

$$P \frac{b}{l} - T = 0 \rightarrow T = P \frac{b}{l}$$

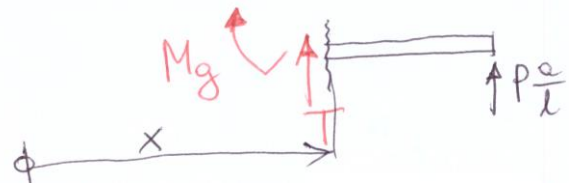


$$\sum M_{pp} = 0:$$

$$-P \frac{b}{l} \cdot x + M_g = 0$$

$$M_g = P \frac{b}{l} \cdot x$$

(II) $x \in (a, l)$



$$\sum F_z = 0:$$

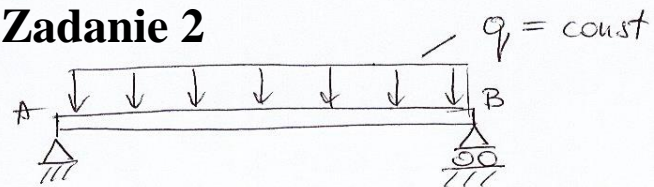
$$T + P \frac{a}{l} = 0 \Rightarrow T = -P \frac{a}{l}$$

$$\sum M_{pp} = 0:$$

$$P \frac{a}{l} \cdot (l - x) - M_g = 0$$

$$M_g = P \frac{a}{l} (l - x)$$

Zadanie 2



Realizacja:

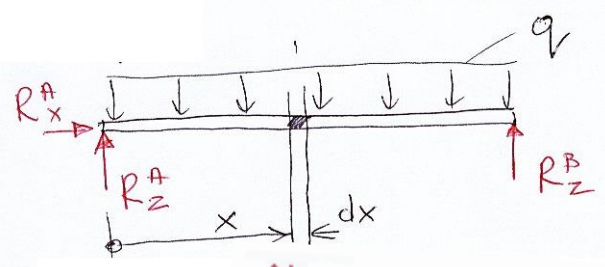
$$\sum M_A = 0: R_B \cdot l - \int_0^l q \, dx \cdot x = 0$$

$$\sum F_z = 0: R_z^A + R_z^B - \int_0^l q \, dx = 0$$

$$R_z^A = q \cdot x \Big|_0^l = \frac{q \cdot l}{2}$$

$$R_B^z = \frac{1}{l} \cdot q \cdot \frac{1}{2} x^2 \Big|_0^l \quad R_B^z = \frac{q \cdot l}{2}$$

$$R_z^A = \frac{q \cdot l}{2}$$



r-wie r-gi: $\sum F_z = 0:$

$$\frac{q \cdot l}{2} - \int_0^x q \, ds - T = 0$$

$\sum M_{pp} = 0:$

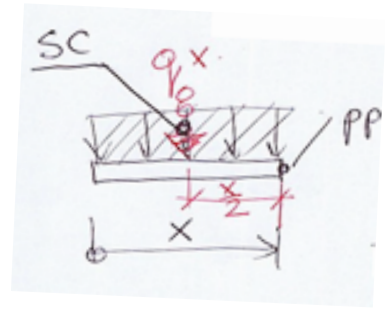
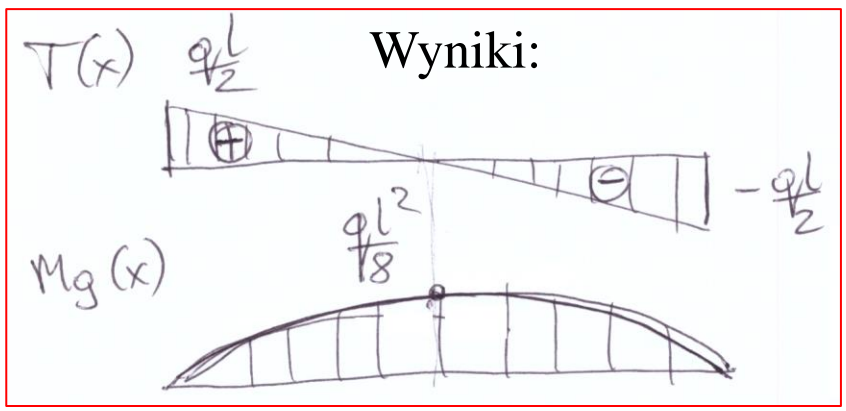
$$T = \frac{q \cdot l}{2} - q \cdot s \Big|_0^x = q \left(\frac{l}{2} - x \right)$$



$$-\frac{q \cdot l}{2} \cdot x + \int_0^x q \, ds \cdot (x-s) + M_g = 0 \rightarrow M_g = \frac{q \cdot l}{2} \cdot x - q \int_0^x (x-s) \, ds$$

$$M_g = \frac{q \cdot l}{2} x - q \left(xs - \frac{1}{2} s^2 \right) \Big|_0^x \rightarrow M_g = \frac{q \cdot l}{2} x - q \left(x^2 - \frac{1}{2} x^2 \right)$$

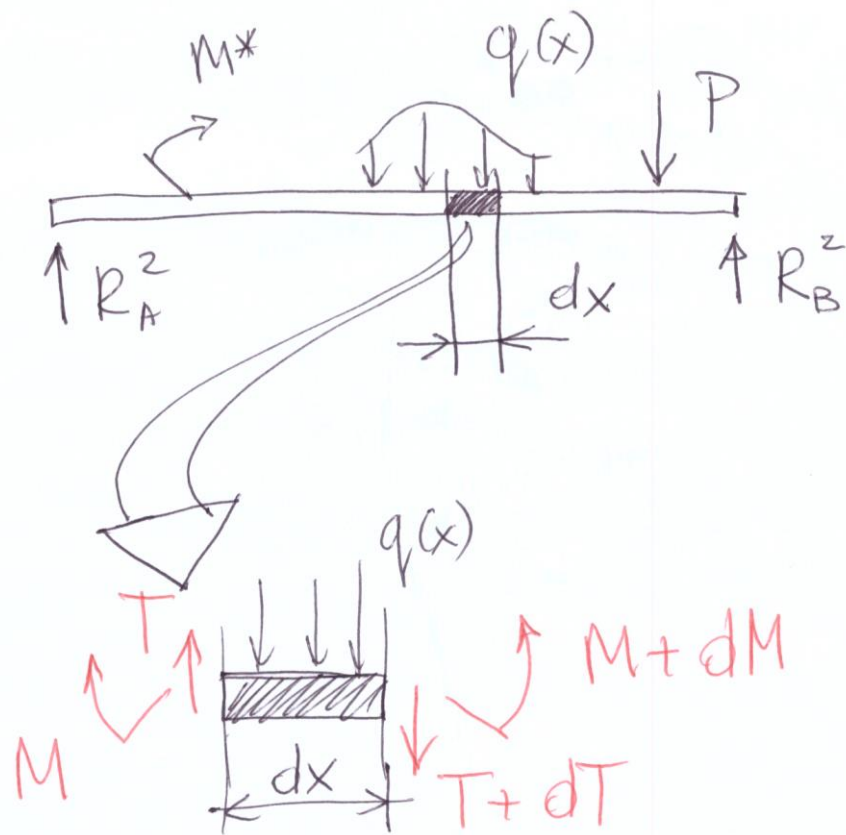
$$M_g = \frac{q \cdot l}{2} x - \frac{1}{2} q x^2$$



graficzne całkowanie

Wypadkowe siła w SC

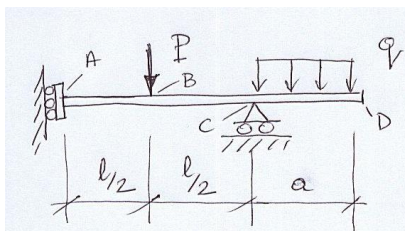
ramię działające



$$q(x) = - \frac{dT(x)}{dx}$$

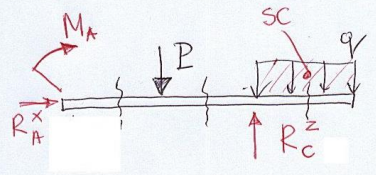
$$T(x) = \frac{dMg(x)}{dx}$$

Zadanie 3



$P = 2 \text{ kN}$
 $q = 10 \frac{\text{kN}}{\text{m}}$
 $l = 1 \text{ m}$
 $a = 0.4 \text{ m}$
 $T(x)$
 $M_g(x)$
 $?$

Reakcje:
 r-nie r-gi:
 $\sum F_x = 0: R_A^x = 0$



$\sum M_c = 0: -M_A + P \cdot \frac{l}{2} - q \cdot a \cdot \frac{a}{2} = 0$

$M_A = \frac{Pl}{2} - \frac{qa^2}{2}$

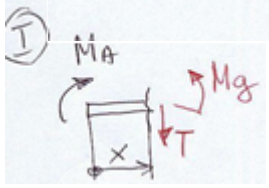
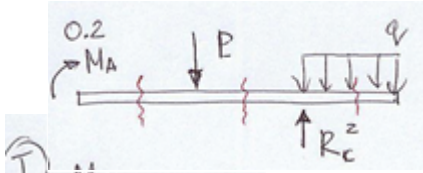
$M_A = \frac{2 \cdot 1}{2} - \frac{10 \cdot 0.4^2}{2}$

$M_A = 1 - 0.8 = 0.2 \text{ kNm}$

$\sum F_z = 0: -P + R_c^z - qa = 0$

$R_c^z = qa + P$

$R_c^z = 10 \cdot 0.4 + 2 = 6 \text{ kN}$



Mysłowe przecięcie:

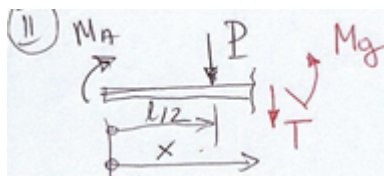
$\sum F_z = 0: -T = 0$

$\sum M_{pp} = 0: -M_A + M_g = 0$

$M_g(x) = M_A$

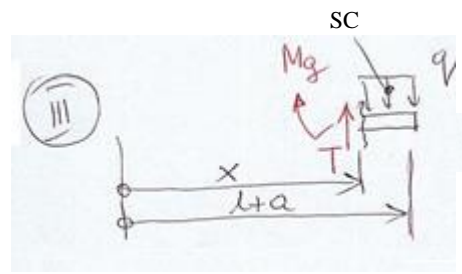
$\sum F_z = 0: -P - T = 0$

$T(x) = -P$



$\sum M_{pp} = 0: -M_A + P(x - \frac{l}{2}) + M_g = 0$

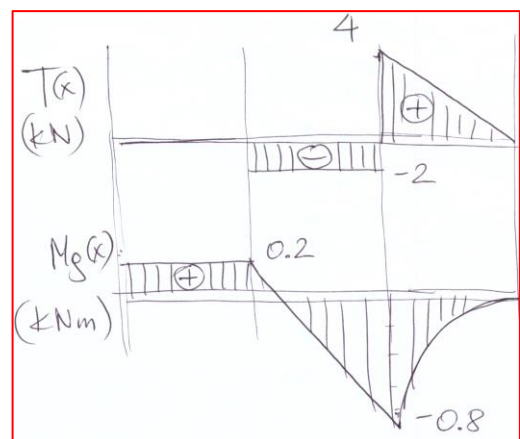
$M_g(x) = M_A + P(\frac{l}{2} - x)$



$\sum F_z = 0: T - q \cdot (l + a - x) = 0$

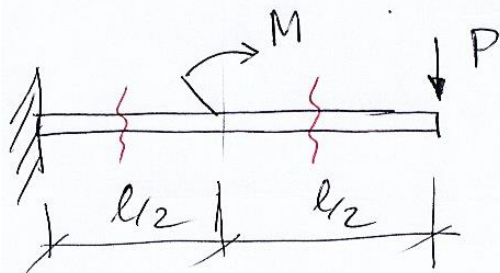
$T(x) = q(l + a - x)$

Wyniki:



$\sum M_{pp} = 0:$
 $-M_g - \frac{q \cdot (l + a - x)^2}{2} = 0$
 $M_g(x) = -q \frac{(l + a - x)^2}{2}$

Zadanie 4



$$\begin{array}{l|l} M = 0.5 \text{ kNm} & M_g(x) \\ P = 1 \text{ kN} & T(x) \\ l = 1 \text{ m} & 2 \end{array}$$

r-woie r-gi:

$$\sum F_z = 0: T - P = 0$$

$$T(x) = P$$

$$\sum M_{pp} = 0: -M_g - P \cdot (l - x) = 0$$

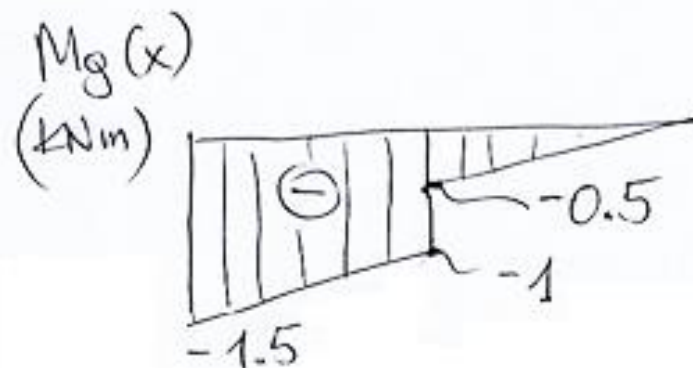
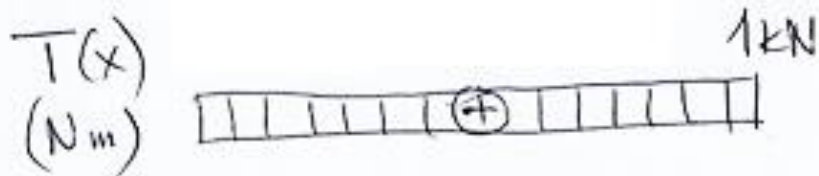
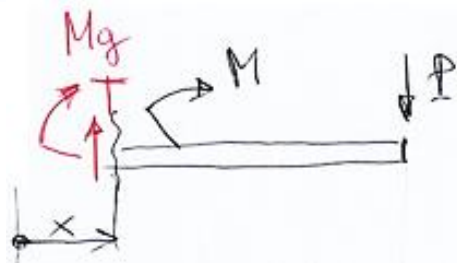
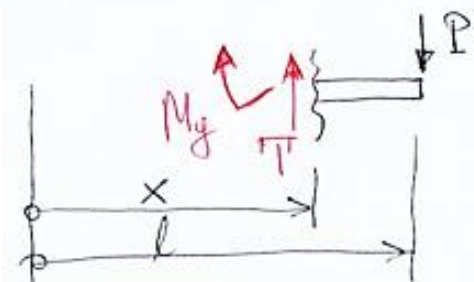
$$M_g(x) = -P(l - x)$$

$$\sum F_z = 0: T - P = 0$$

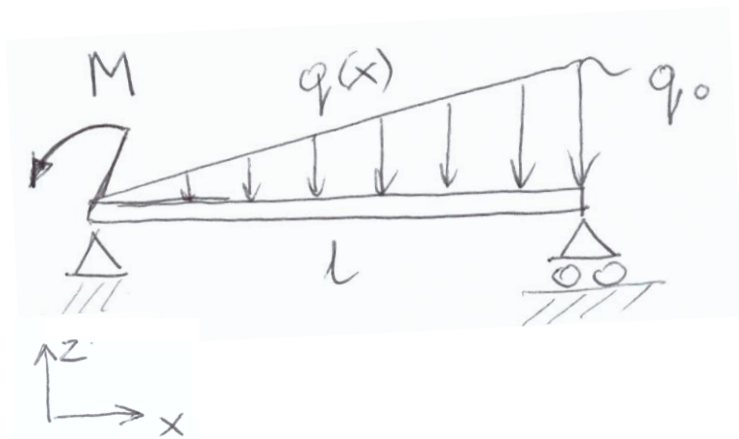
$$T(x) = P$$

$$\sum M_{pp} = 0: -M_g - M - P(l - x) = 0$$

$$M_g(x) = -P(l - x) - M$$



Zadanie 5

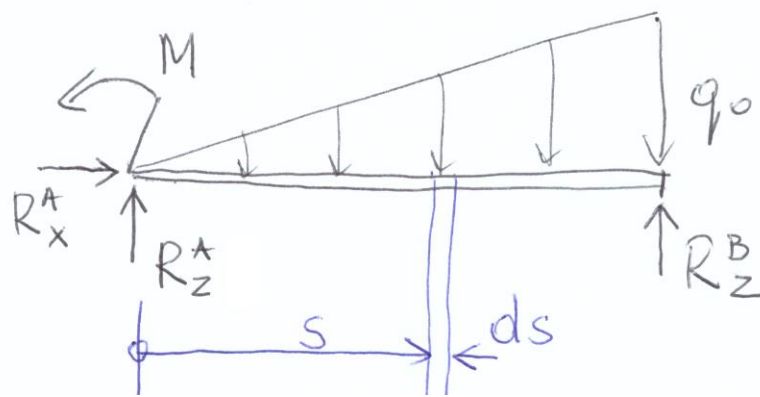


$$q_0 = g \frac{\text{kN}}{\text{m}}$$

$$M = 1,25 \text{ kNm}$$

$$l = 1 \text{ m}$$

$$q(x) = \frac{q_0}{l} \cdot x$$



Reakcje: $\sum M_A = 0$:

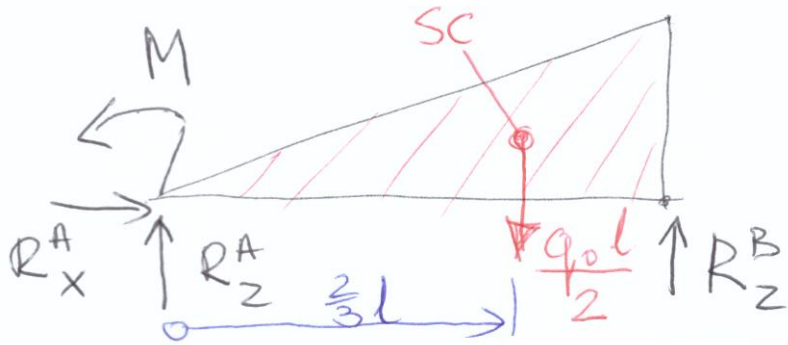
$$-\int_0^l s \cdot q(s) \cdot ds + R_z^B \cdot l + M = 0$$

$$R_z^B = \int_0^l \frac{q_0}{l^2} \cdot s^2 ds - \frac{M}{l}$$

$$R_z^B = \frac{q_0}{l^2} \frac{1}{3} l^3 - \frac{M}{l}$$

$$R_z^B = \frac{q_0 l}{3} - \frac{M}{l}$$

Prościej (graficznie)



$$\sum M_A = 0:$$

$$- \frac{q_0 l}{2} \cdot \frac{2}{3} l + R_z^B \cdot l + M = 0$$

$$R_z^B = \frac{q_0 l}{3} - \frac{M}{l} = 1,75 \text{ kN}$$

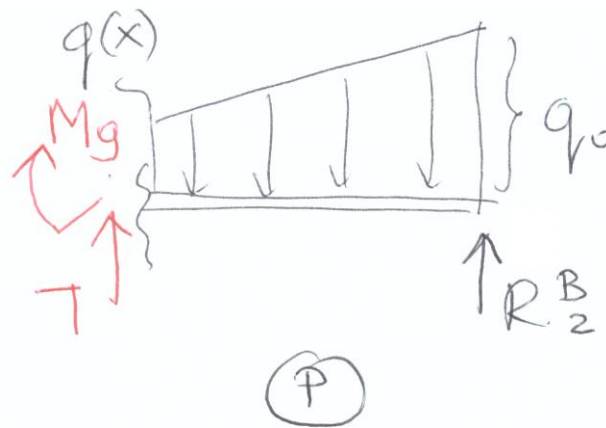
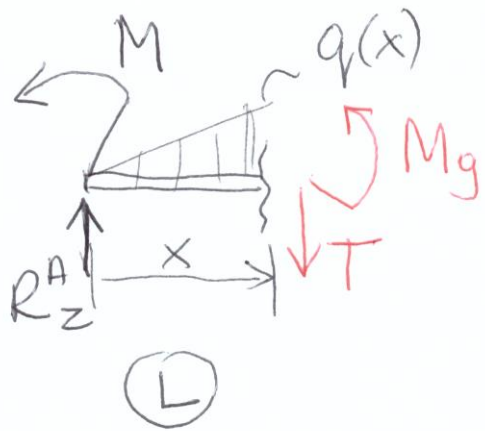
$$\sum M_B = 0:$$

$$\frac{q_0 l}{2} \cdot \frac{1}{3} l - R_z^A \cdot l + M = 0$$

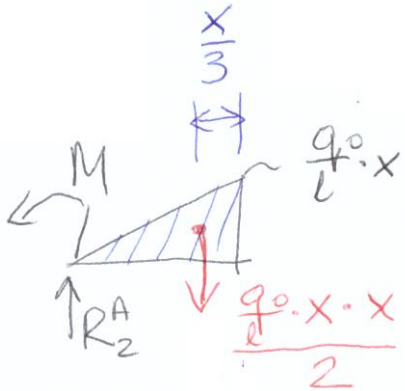
$$R_z^A = \frac{q_0 l}{6} + \frac{M}{l} = 2,75 \text{ kN}$$

$$\sum F_x = 0:$$

$$R_x^A = 0$$



$$\Sigma F_z = 0: -T + R_z^A - \int_0^x q(s) ds = 0$$



$$T = \boxed{\frac{q_0 l}{6} + \frac{M}{l}} - \underbrace{\frac{\frac{q_0 \cdot x \cdot x}{2}}{2}}_{\text{graficune}} = \boxed{\frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 x^2}{2l}}$$

$$\Sigma M_{pp} = 0: M_g + \int_0^x (x-s) q(s) \cdot ds - R_z^A \cdot x + M = 0$$

$$M_g = \boxed{\left(\frac{q_0 l}{6} + \frac{M}{l}\right) \cdot x - M} - \underbrace{\frac{\frac{q_0}{l} x \cdot x}{2} \cdot \frac{x}{3}}_{\text{graficune}}$$

$$M_g = -\frac{q_0 x^3}{6l} + \left(\frac{q_0 l}{6} + \frac{M}{l}\right) \cdot x - M$$

$$T = \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 x^2}{2l}$$

$$T(0) = \frac{q_0 l}{6} + \frac{M}{l} = \frac{9 \cdot 1}{6} + \frac{1,25}{1} = 2,75 \text{ kN} = R_2^+$$

$$T(l) = \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 l}{2} = -1,75 \text{ kN} = -R_2^B$$

$$T(x_0) = 0$$

$$\rightarrow \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 x_0^2}{2l} = 0$$

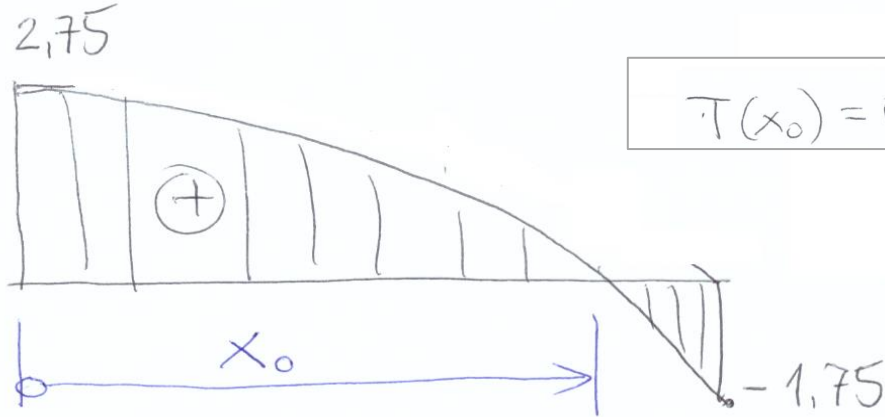
$$x_0 = \pm \sqrt{\left(\frac{q_0 l}{6} + \frac{M}{l}\right) \cdot \frac{2l}{q_0}}$$

$$x_0 = \pm \sqrt{\left(\frac{9 \cdot 1}{6} + \frac{1,25}{1}\right) \cdot \frac{2 \cdot 1}{9}}$$

$$x_0 = 0,782 \text{ m}$$

T

[kN]



$$M_g = -\frac{q_0 x^3}{6l} + \left(\frac{q_0 l}{6} + \frac{M}{l}\right) \cdot x - M$$

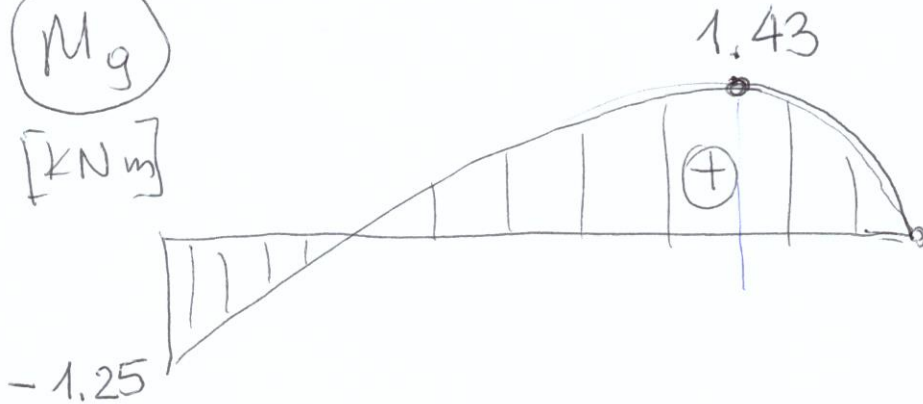
$$M_g(0) = -M = -1,25 \text{ kNm}$$

$$M_g(l) = 0$$

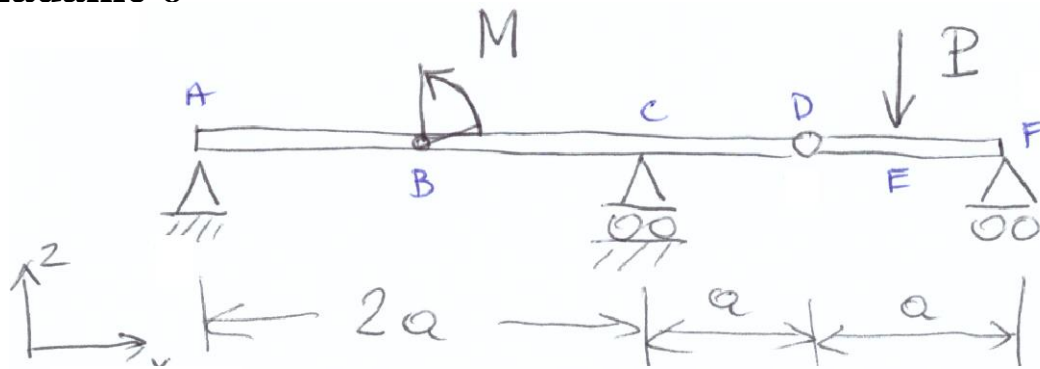
$$M_g(x_0) = 1,43 \text{ kNm}$$

M_g

[kNm]



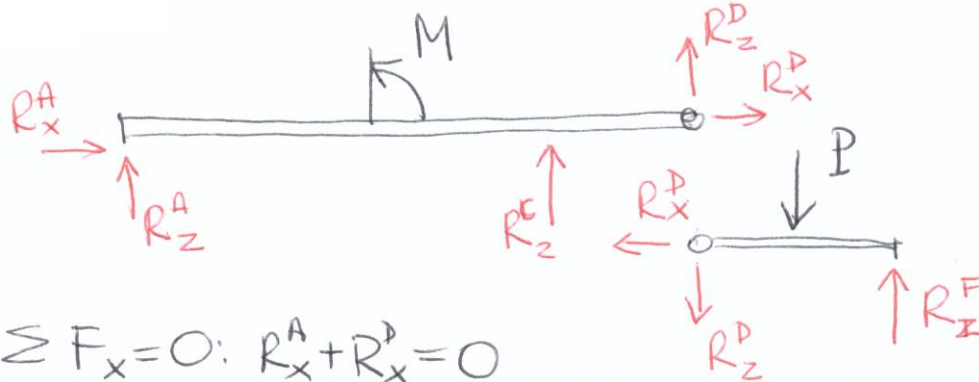
Zadanie 6



$$M = 1 \text{ kNm}$$

$$P = 1 \text{ kN}$$

$$a = 1 \text{ m}$$



$$\sum F_x = 0: R_x^A + R_x^D = 0$$

$$R_x^A = 0$$

$$\sum M_A = 0: M + R_z^C \cdot 2a + R_z^D \cdot 3a = 0$$

$$R_z^C = -\frac{M}{2a} + \frac{3}{4}P$$

$$\sum F_z = 0: R_z^A + R_z^C + R_z^D = 0 \rightarrow R_z^A = \frac{M}{2a} - \frac{3}{4}P + \frac{P}{2} \rightarrow R_z^A = \frac{M}{2a} - \frac{P}{4}$$

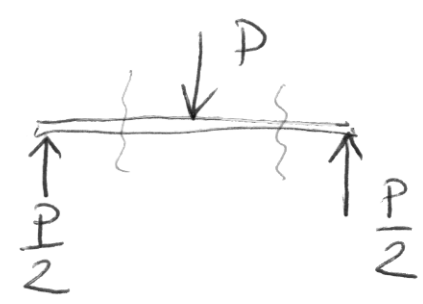
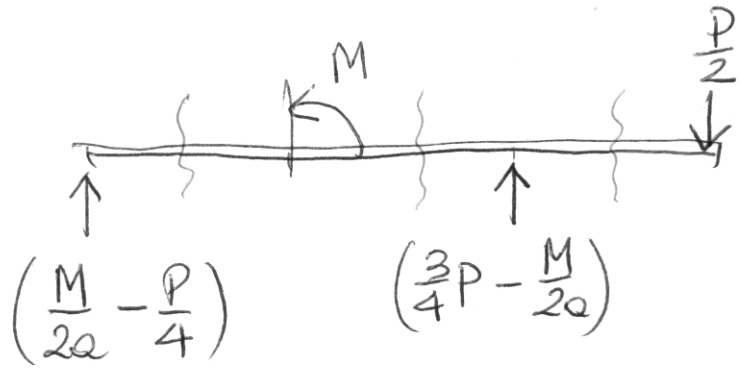
$$\sum F_x = 0: R_x^D = 0$$

$$\sum M_D = 0: -P \cdot \frac{a}{2} + R_z^F \cdot a = 0$$

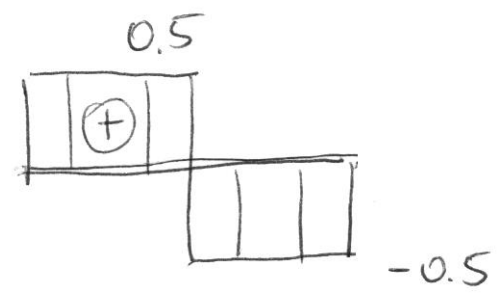
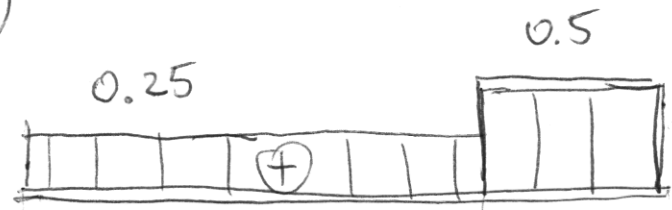
$$R_z^F = \frac{P}{2}$$

$$\sum F_z = 0: R_z^D = -P + R_z^F$$

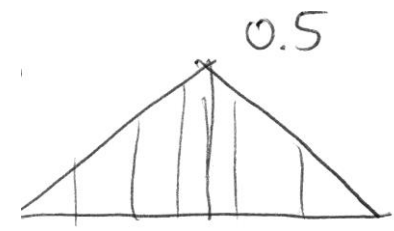
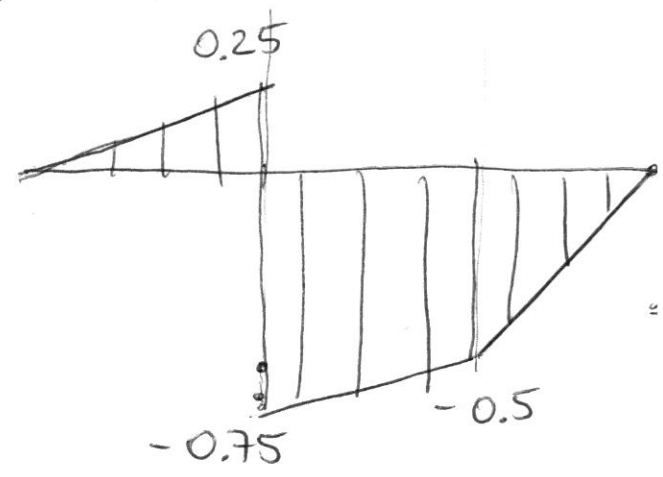
$$R_z^D = -\frac{P}{2}$$



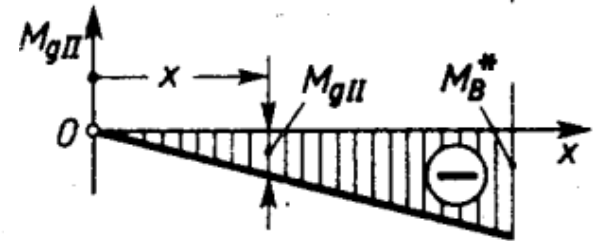
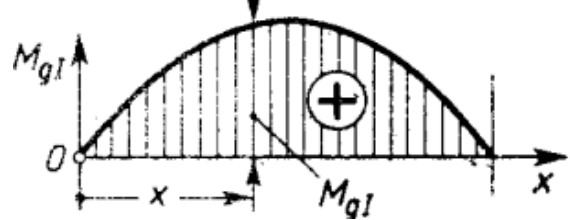
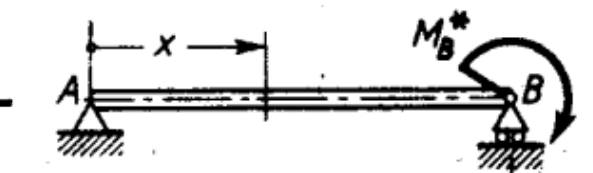
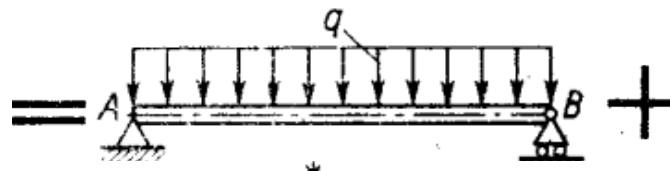
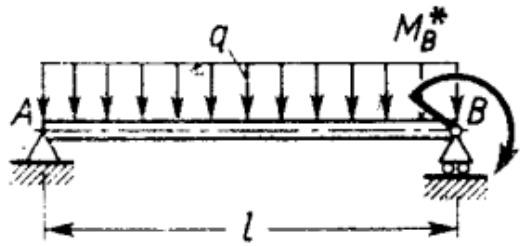
(T)



(M_g)

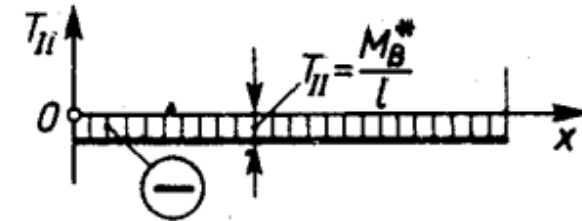
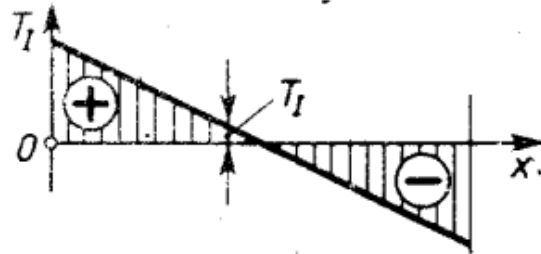


Niektóre ułatwienia obliczeń

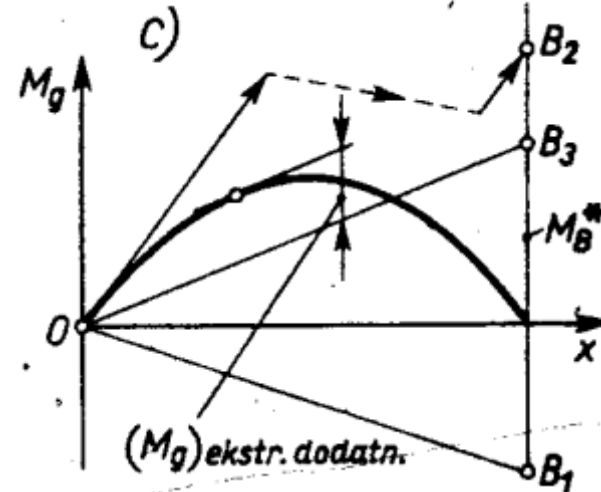
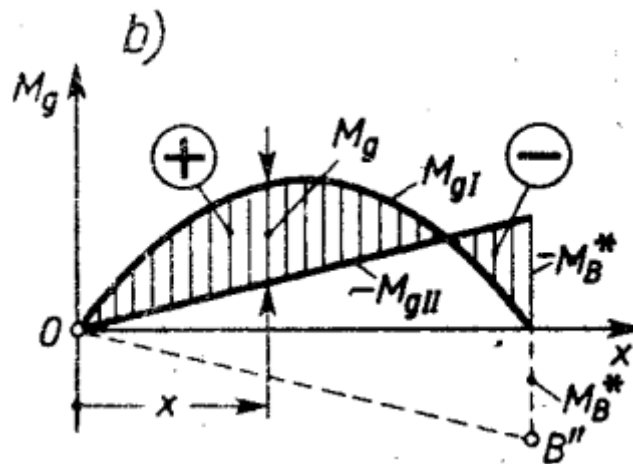
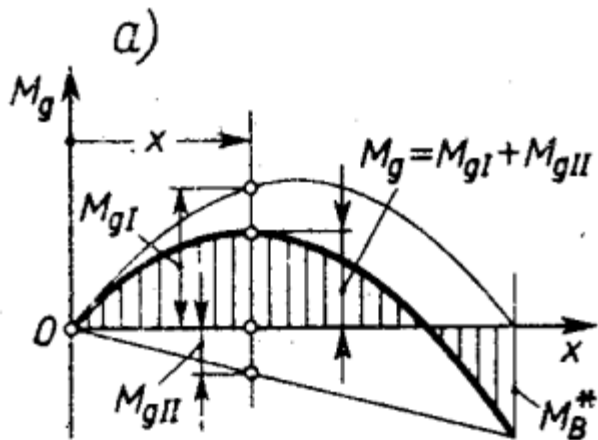


$$M_g = M_{gI} + M_{gII}$$

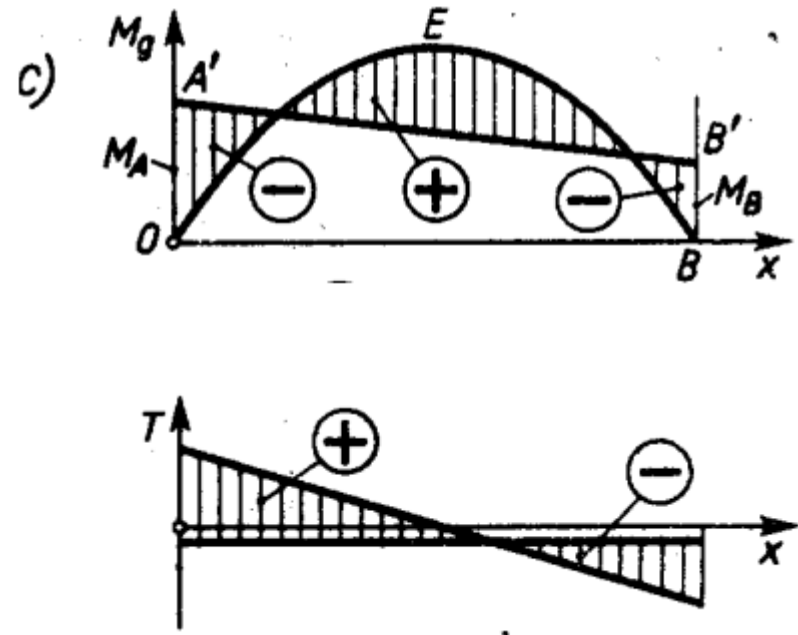
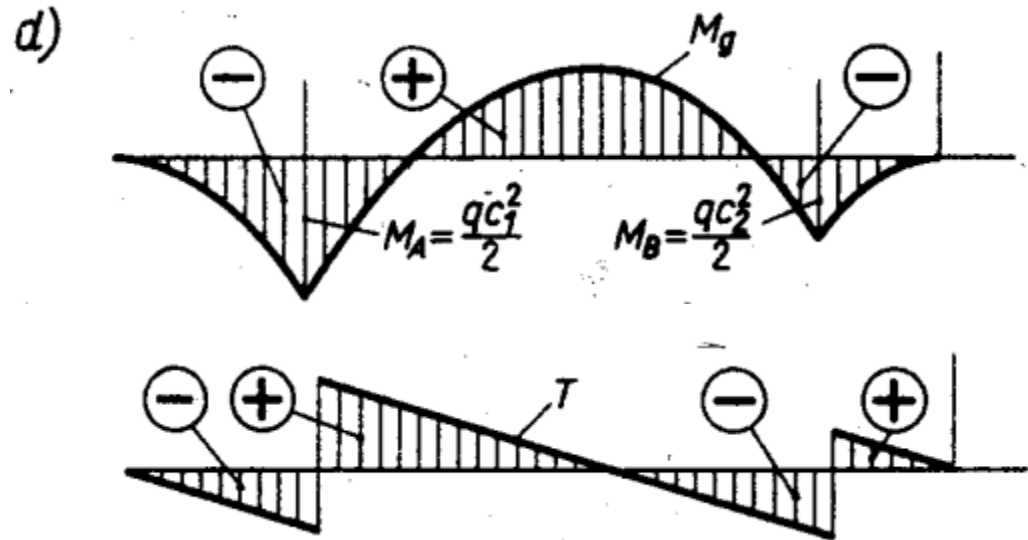
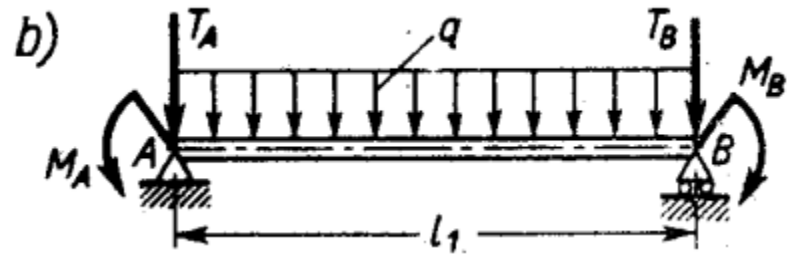
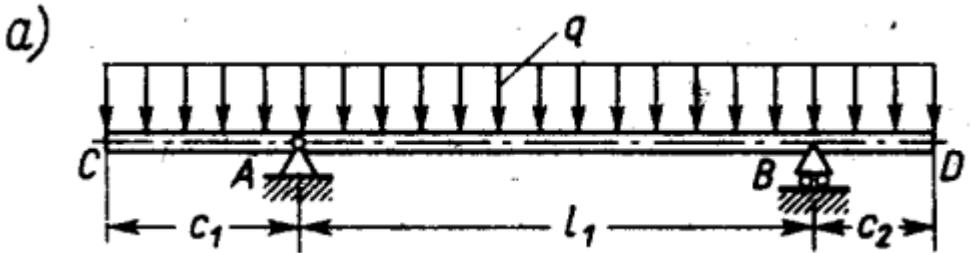
$$T = T_I + T_{II}$$



Zastosowanie zasady superpozycji do określania M_g i T

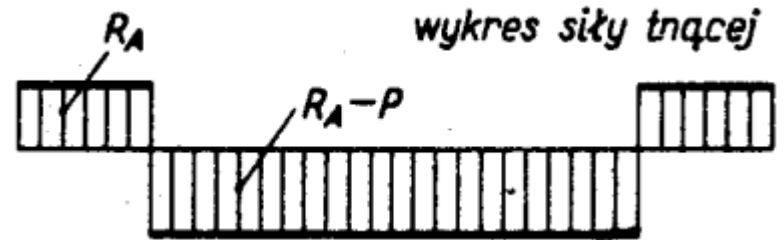
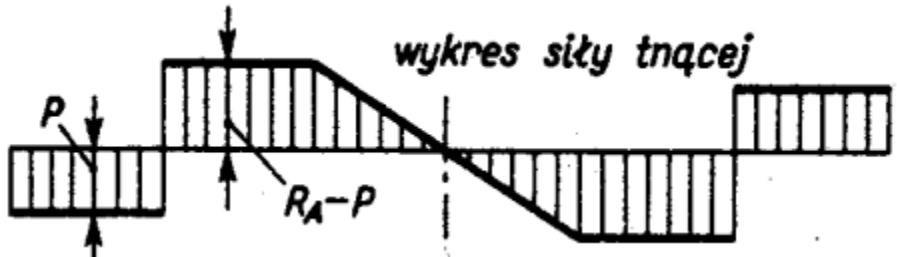
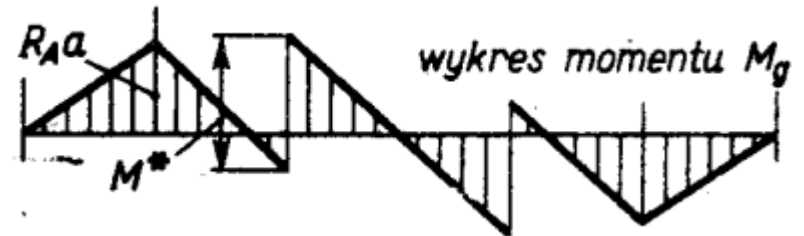
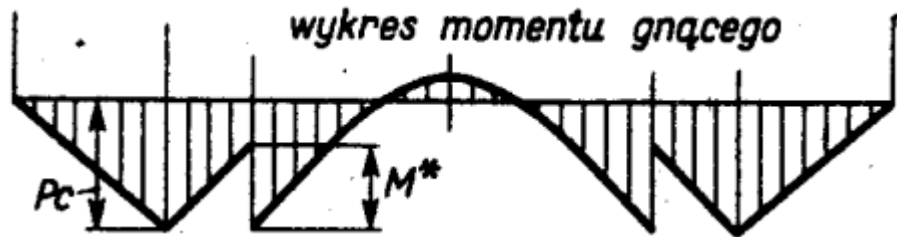
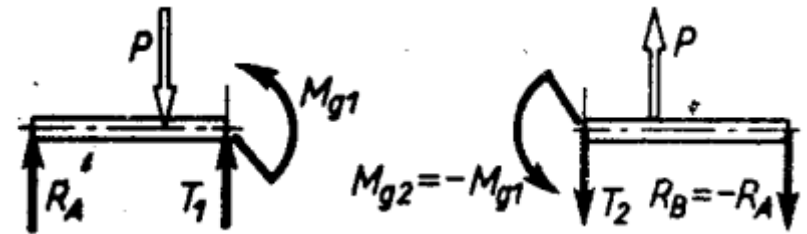
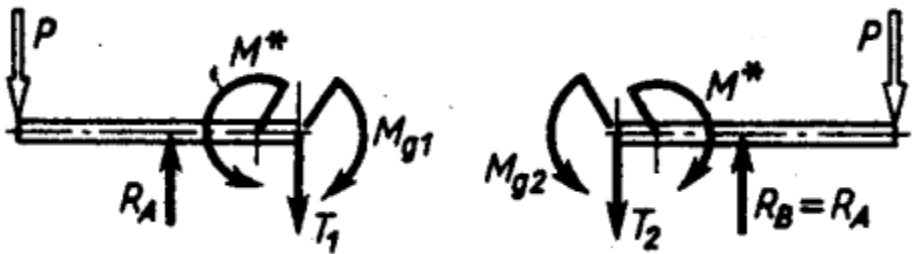
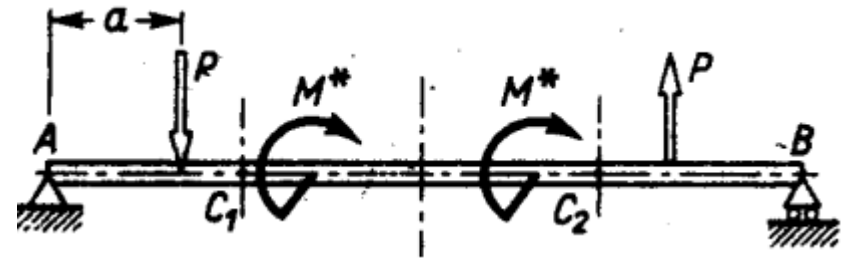
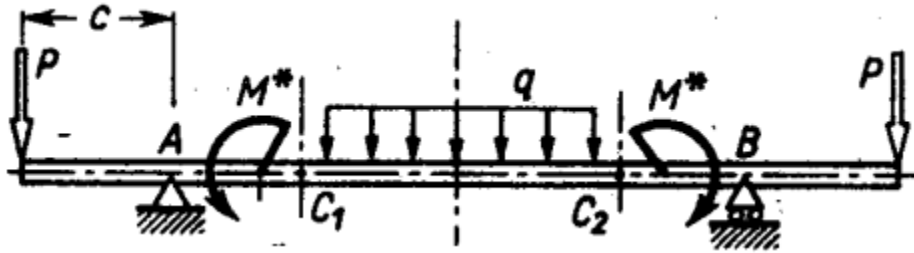


Niektóre ułatwienia obliczeń



Przykład zastosowania zasady superpozycji

Niektóre ułatwienia obliczeń



Wykresy T i M_g przy obciążeniu symetrycznym

Wykresy T i M_g przy obciążeniu antysymetrycznym